# 478: Midterm

**1. [Graphs, Relations, BDDs] (3 x 7 = 21 Points)**

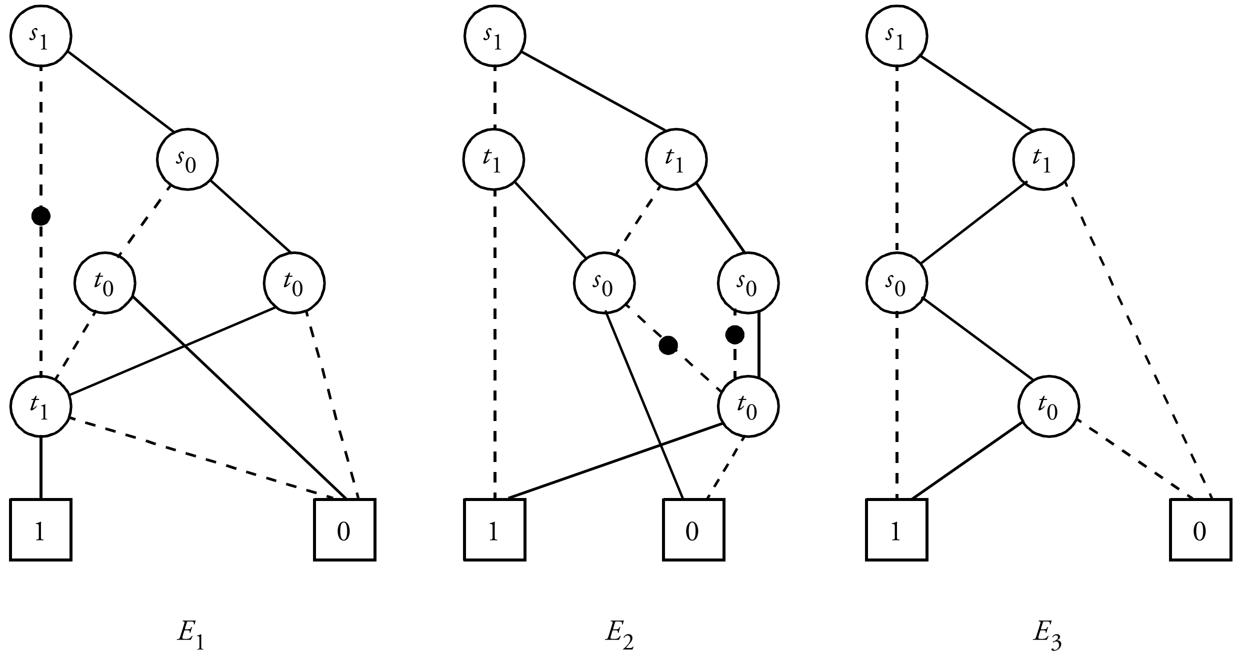
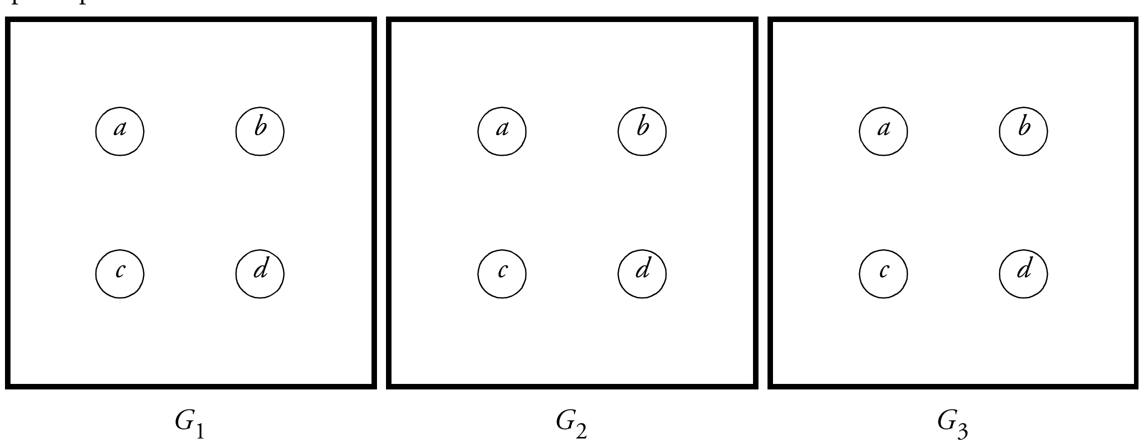



Figure 1. **BDDs for Graph Edge Functions**

The BDDs in Figure 1 represent the edge membership functions for three graphs $G_1$, $G_2$, and $G_3$, where the $s = s_1 s_0$ and $t = t_1 t_0$ variables denote the source and target vertices for the edges. Using the vertex labels $a$, $b$, $c$, and $d$ encoded as $a = 00$, $b = 01$, $c = 10$, $d = 11$:

a. Draw each of the three graphs in the square boxes below.

b. What type of binary relation (equivalence, partial order, etc.) on the set $\{a, b, c, d\}$ does each of these graphs represent?



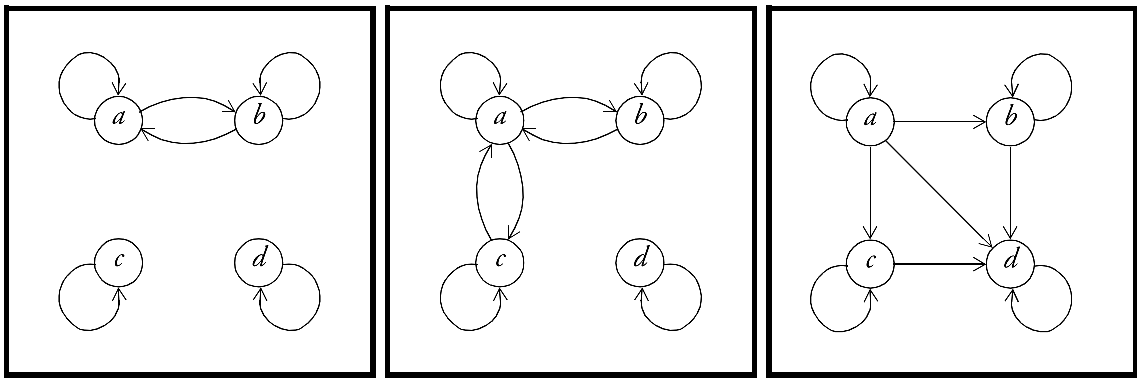

Relation Type __________ __________ __________

## Work Sheet for Problem 1

The functions represented by each of the BDDs are:

$$E_1 : \overline{s_1}\,\overline{t_1} + s_1\overline{s_0}t_1\overline{t_0} + s_1 s_0 t_1 t_0$$
$$E_2 : \overline{s_1}\,\overline{t_1} + \overline{s_1}\,\overline{s_0}t_1\overline{t_0} + s_1\overline{s_0}\,\overline{t_1}\,\overline{t_0} + s_1\overline{s_0}t_1\overline{t_0} + s_1 s_0 t_1 t_0$$
$$E_3 : \overline{s_1}\,\overline{s_0} + \overline{s_1}t_0 + s_1\overline{s_0}t_1 + s_1 t_1 t_0$$

The corresponding graphs are:




$G_1$ $G_2$ $G_3$

| Relation Type | $G_1$ | $G_2$ | $G_3$ |
|---|---|---|---|
| | reflexive, symmetric, transitive $\longrightarrow$ Equivalence | reflexive, symmetric, not transitive $\longrightarrow$ Compatibility | reflexive, antisymmetric, transitive $\longrightarrow$ Partial order |

**2. [Solving Boolean Equations] (19 Points)**

The function table for an SR latch (shown at right) expresses next state $Q^+$ in terms of the set $S$ and reset $R$ inputs, as well as present state $Q$.

| $S$ | $R$ | $Q^+$ |
|---|---|---|
| 0 | 0 | $Q$ |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | Invalid |

a. (6 Points) Write a system of simultaneous algebraic equations that completely characterizes the behavior specified by this function table. (*Hint: you need one equation for $Q^+$ and another to model the constraint on $S$ and $R$.*)

b. (8 Points) Solve, symbolically, the system of equations in (a) for both $S$ and $R$ as functions of $Q$ and $Q^+$. Express the solutions as function intervals and as truth tables. Show all steps of your derivation for full credit.

c. (5 Points) Solve, symbolically, for $Q$ as a function of $S$ and $R$. Interpret the result that you get.

## Work Sheet for Problem 2

a. From the function table, the next-state equation can be readily expressed as $Q^+ = S'R'Q + SR' = R'(S+Q)$. Another, perhaps more familiar, expression is $Q^+ = S + R'Q$. The difference between the two expressions stems from the treatment of the invalid combination $SR = 11$; the first expression sets $Q^+$ to 0 whereas the second expression sets it to 1. Both of these choices, of course, are valid as long as we enforce the constraint $SR = 0$.

b. We'll illustrate the solution using the first expression for $Q^+$. The result will be identical when the other expression is used.

$$[Q^+ = R'(S+Q)] \text{ and } [SR = 0]$$

$$[Q^+ \oplus R'(S+Q) = 0] \text{ and } [SR = 0]$$

$$[Q^+ \oplus R'(S+Q)] + SR = 0$$

Let $F(Q, Q^+, S, R) \equiv [Q^+ \oplus R'(S+Q)] + SR$. Then $F = 0$.

Eliminate $R$: $\forall R \cdot F = [Q^+ \oplus (S+Q)][(Q^+ \oplus 0) + S] = Q'Q^+S' + (Q^+)'S = 0$

This leads to $S \in [Q'Q^+, Q^+]$.

Eliminate $S$: $\forall S \cdot F = [Q^+ \oplus R'Q][(Q^+ \oplus R') + R] = Q(Q^+)'R' + Q^+R = 0$

This leads to $R \in [Q(Q^+)', (Q^+)']$.

In tabular form:

| $Q$ | $Q^+$ | $S$ Interval | | $R$ Interval | | $S$ | $R$ |
|---|---|---|---|---|---|---|---|
| | | $Q'Q^+$ | $Q^+$ | $Q(Q^+)'$ | $(Q^+)'$ | | |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | X |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | X | 0 |

c. Eliminate $Q^+$: Let $G \equiv \forall Q^+ \cdot F = [(R'(S+Q)) + SR][(R'(S+Q))' + SR] = SR = 0$.

Thus, $G_Q = G_{Q'} = SR = 0$, and $Q \in [0, 1]$. This means that $Q$ is not a function of either $S$ or $R$.

**3. [Interval Boolean Algebra] (20 Points)**

a. (15 Points) Let $F = [\lfloor F \rfloor, \lceil F \rceil]$ and $G = [\lfloor G \rfloor, \lceil G \rceil]$ be two intervals in the lattice of $n$-variable switching functions. Prove that the logical OR of these two intervals is also an interval given by the following expression:

$$F + G = [\lfloor F \rfloor + \lfloor G \rfloor, \lceil F \rceil + \lceil G \rceil].$$

Show all steps of your proof for full credit.

b. (5 Points) Let $F(a, b, c) = [b'c + a'bc', b'c + a'b]$ and $G(a, b, c) = [a'b' + abc, a' + bc]$. Fill in the truth table for the partially specified function $F + G$.

| $a$ | $b$ | $c$ | $F + G$ |
|---|---|---|---|
| 0 | 0 | 0 | |
| 0 | 0 | 1 | |
| 0 | 1 | 0 | |
| 0 | 1 | 1 | |
| 1 | 0 | 0 | |
| 1 | 0 | 1 | |
| 1 | 1 | 0 | |
| 1 | 1 | 1 | |

## Work Sheet for Problem 3

a. Let $H = F + G$. Without loss of generality, let $F = \{f_1, \ldots, f_m\}$ and $G = \{g_1, \ldots, g_n\}$. Note that $\lfloor F \rfloor = \prod_{1 \le i \le m} f_i$ and $\lceil F \rceil = \sum_{1 \le i \le m} f_i$. Since $F$ is an interval, $\lfloor F \rfloor \in F$ and $\lceil F \rceil \in F$. A similar argument applies to $G$, Using functional composition rule #1, we can express the function set $H$ as:

$$H = \bigcup_{\substack{1 \le i \le m \\ 1 \le j \le n}} \{f_i + g_j\}$$. Thus,

$$\begin{aligned}
\lfloor H \rfloor &= \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (f_i + g_j) \\
&= (f_1 + g_1)(f_1 + g_2)\ldots(f_1 + g_n)\ldots(f_m + g_1)(f_m + g_2)\ldots(f_m + g_n) \\
&= (f_1 + g_1 \ldots g_n)\ldots(f_m + g_1 \ldots g_n) \\
&= (f_1 \ldots f_m + g_1 \ldots g_n) \\
&= \lfloor F \rfloor + \lfloor G \rfloor
\end{aligned}$$

where the distributive law was applied to simplify the $m \times n$-term POS expression to a two-term SOP expression. Since $\lfloor F \rfloor \in F$ and $\lfloor G \rfloor \in G$, we conclude that $\lfloor H \rfloor \in H$.

Next,

$$\begin{aligned}
\lceil H \rceil &= \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} (f_i + g_j) \\
&= (f_1 + g_1) + (f_1 + g_2) + \ldots + (f_1 + g_n) + \ldots + (f_m + g_1) + (f_m + g_2) + \ldots + (f_m + g_n) \\
&= (f_1 + \ldots + f_m + g_1 + \ldots + g_n) \\
&= \lceil F \rceil + \lceil G \rceil
\end{aligned}$$

where we used idempotency to simplify the expression. Again, $\lceil H \rceil \in H$. Thus, $H$ is the interval specified by $[\lfloor F \rfloor + \lfloor G \rfloor, \lceil F \rceil + \lceil G \rceil]$.

Alternative Solution:

First, show that $(w \le x) \cdot (y \le z) \to (w + y \le x + z)$ is a valid identity.

$$\begin{aligned}
(w \le x) \cdot (y \le z) \to (w + y \le x + z) &= \overline{(w' + x) \cdot (y' + z)} + \overline{(w + y)} + x + z \\
&= wx' + yz' + w'y' + x + z \\
&= (wx' + x) + (yz' + z) + w'y' \\
&= w + x + y + z + w'y' \\
&= (w + w'y') + x + y + z \\
&= w + y' + x + y + z \\
&= 1
\end{aligned}$$

Next, express the lower bound of $H$ in a form that allows application of the above identity:

$$\lfloor H \rfloor = \prod_{\substack{f \in F \\ g \in G}} (f+g) = (\lfloor F \rfloor + \lfloor G \rfloor) \prod_{\substack{f \in F \\ g \in G}} (f+g)$$

The above is true because of idempotency and the fact that $\lfloor F \rfloor$ and $\lfloor G \rfloor$ are elements of $F$ and $G$. Since $\lfloor F \rfloor \le f$ and $\lfloor G \rfloor \le g$ for all $f \in F$ and $g \in G$, we can now use the identity we proved above to state that $(\lfloor F \rfloor + \lfloor G \rfloor) \le (f+g)$ which is equivalent to $(\lfloor F \rfloor + \lfloor G \rfloor)(f+g) = (\lfloor F \rfloor + \lfloor G \rfloor)$. Thus, $\lfloor H \rfloor = (\lfloor F \rfloor + \lfloor G \rfloor)$.

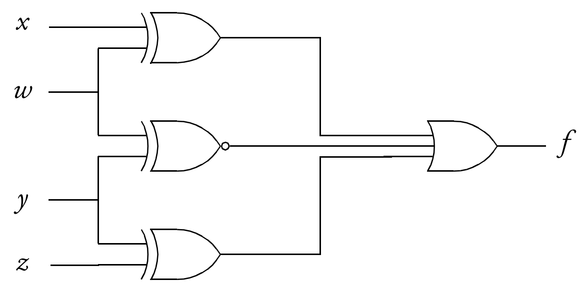
The upper bound is determined by a similar argument.

b. By inspection, the truth table is:

| $a$ | $b$ | $c$ | $F+G$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | d |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

**4. [Prime Implicants] (20 Points)**

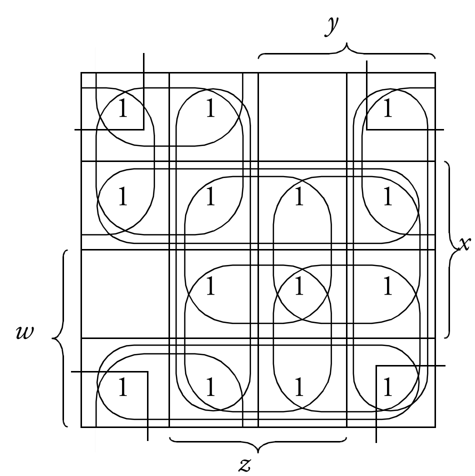
Using any method of your choice, determine the complete set of prime implicants of the function specified by the circuit shown here and indicate which ones are essential.




There are several approaches to solving this problem. Since there are only four variables, a K-map might be a good way to proceed. An easy way to "fill the map" would be to cofactor $f$ with respect to two of the variables, say $w$ and $x$. By inspection, these cofactors are

$$\begin{aligned} f_{w'x'} &= 0 + y' + (y \oplus z) = y' + z' \\ f_{w'x} &= 1 \\ f_{wx'} &= 1 \\ f_{wx} &= 0 + y + (y \oplus z) = y + z \end{aligned}$$

yielding the desired map.




There is a total of 12 prime implicants:

$$w'x,\ w'y',\ w'z',\ wx',\ wy,\ wz,\ x'y',\ x'z',\ xy,\ xz,\ y'z,\ yz'$$

An alternative algebraic approach would start from the following SOP expression for $f$ derived from the circuit:

$$f = w'x + wx' + w'y' + wy + y'z + yz'$$

The prime implicants can now be systematically generated using the iterative consensus algorithm.

5. **[Set Covering] (20 Points)**

Consider the set covering problem described by the following constraint matrix with associated column costs:

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
|---|---|---|---|---|---|---|---|
| C1 | 1 | 1 | | | | 1 | |
| C2 | 1 | | | 1 | | 1 | |
| C3 | | 1 | 1 | | 1 | | |
| C4 | | | | 1 | | 1 | 1 |
| Cost | 2 | 3 | 4 | 1 | 2 | 1 | 4 |

a. (4 Points) Assuming that the weight of a row is the number of columns covering it, determine a maximal independent set of constraints and the associated lower bound on the cost of the solution using the MIS_QUICK algorithm. Break ties in favor of the lower-numbered constraint.

Answer:

MIS = ______________________________

Lower Bound = ______________________________

Solution: (2 + 2 pts):

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | w |
|---|---|---|---|---|---|---|---|---|
| C1 | 1 | 1 | | | | 1 | | 3 |
| C2 | 1 | | | 1 | | 1 | | 3 |
| C3 | | 1 | 1 | | 1 | | | 3 |
| C4 | | | | 1 | | 1 | 1 | 3 |
| Cost | 2 | 3 | 4 | 1 | 2 | 1 | 4 | |

MIS = {C1}, LB = min(2, 3, 1) = 1

b. (6 Points) Repeat part a assuming that the weight of a row is the sum of its column counts.

Answer:

MIS = ______________________________

Lower Bound = ______________________________

Solution: (4 + 2 pts)

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | w |
|---|---|---|---|---|---|---|---|---|
| C1 | 1 | 1 | | | | 1 | | 7 |
| C2 | 1 | | | 1 | | 1 | | 7 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| C3 | | 1 | 1 | | 1 | | | 4 |
| C4 | | | | 1 | | 1 | 1 | 6 |
| Cost | 2 | 3 | 4 | 1 | 2 | 1 | 4 | |

MIS = {C3}

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | w |
|---|---|---|---|---|---|---|---|---|
| C2 | 1 | | | 1 | | 1 | | 5 |
| C4 | | | | 1 | | 1 | 1 | 5 |
| Cost | 2 | 3 | 4 | 1 | 2 | 1 | 4 | |

MIS = {C3, C2}, LB = min(3, 4, 2) + min(2, 1, 1) = 2 + 1 = 3

c. (6 Points) What is the optimal solution to this set covering problem and what is its cost?

Answer:

Optimal Solution: ______________________________

Optimal Solution Cost = ______________________________

Solution: (4 + 2 pts):

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
|---|---|---|---|---|---|---|---|
| C1 | 1 | 1 | | | | 1 | |
| C2 | 1 | | | 1 | | 1 | |
| C3 | | 1 | 1 | | 1 | | |
| C4 | | | | 1 | | 1 | 1 |
| Cost | 2 | 3 | 4 | 1 | 2 | 1 | 4 |

B dominates C and is cheaper; remove C (or E is equivalent to C and is cheaper)

F dominates D and is of equal cost; remove D

F dominates G and is cheaper; remove G

F dominates A and is cheaper; remove A

| | *B* | *E* | *F* |
|---|---|---|---|
| C1 | 1 | | 1 |
| C2 | | | 1 |
| C3 | 1 | 1 | |

| | | | |
|---|---|---|---|
| C4 | | | 1 |
| Cost | 3 | 2 | 1 |

F is essential for C4, and covers C1 and C2. C3 can be covered by B or E, but E is cheaper. Hence Solution is E = F = 1, and A = B = C = D = G = 0. Cost = 2+1 = 3.

NOTE: The points in this problem add up to only 16 instead of 20 (we made a mistake!) Your grade reflects the addition of 4 extra points to bring the total to 20.