Homework Assignment #2 Solutions

1. **[XORs] (15 Points)**

Consider a (not necessarily two-valued) Boolean algebra B. Which of the following properties hold for the exclusive OR operator defined by $x \oplus y = x'y + xy'$ for any elements $x, y \in B$?

a. Commutativity

$x \oplus y = x'y + xy'$	by definition of $\oplus$
$= xy' + x'y$	commutativity of +
$= y'x + yx'$	commutativity of *
$= y \oplus x$	definition of $\oplus$

b. Associativity

$(x \oplus y) \oplus z$	$x \oplus (y \oplus z)$
$(xy' + x'y)z' + (xy' + x'y)'z$	$x(yz' + y'z)' + x'(yz' + y'z)$
$xy'z' + x'yz' + x'y'z + xyz$	$xy'z' + xyz + x'yz' + x'y'z$
$xyz + xy'z' + x'yz' + x'y'z$	$xyz + xy'z' + x'yz' + x'y'z$

c. Distributivity over AND and vice versa

Checking the distributivity of AND over $\oplus$

$x \cdot (y \oplus z)$	$(x \cdot y) \oplus (x \cdot z)$
$x(yz' + y'z)$	$(xy)(xz)' + (xy)'(xz)$
$xyz' + xy'z$	$(xy)(x' + z') + (x' + y')xz$
$xyz' + xy'z$	$xyz' + xy'z$

Now, to check the distributivity of $\oplus$ over AND, it happens that there is a counterexample for two valued Boolean algebra

$x \oplus (y \cdot z)$	$(x \oplus y) \cdot (x \oplus z)$
If we pick $x = 1, y = 0, z = 1$	
$1 \oplus (0 \cdot 1)$	$(1 \oplus 0) \cdot (1 \oplus 1)$
1	0

2. **[More XORs] (20 Points)**

For each of the following, prove or show a counter example:

a. If $A \oplus B = 0$, then $A = B$

Solution: $(A \oplus B = 0) \Leftrightarrow (A'B + AB' = 0) \Leftrightarrow (A'B = 0) \wedge (AB' = 0) \Leftrightarrow (B \le A) \wedge (A \le B) \Leftrightarrow (A = B)$

Alternatively: $(A \oplus B = 0) \rightarrow (A = B) = (A \oplus B) + (A \oplus B)' = 1$

b. If $A \oplus C = B \oplus C$, then $A = B$

Solution:

$A \oplus C = B \oplus C$	given
$(A \oplus C) \oplus (B \oplus C) = 0$	from part a
$(A \oplus B) \oplus (C \oplus C) = 0$	associativity
$(A \oplus B) \oplus 0 = 0$	$x \oplus x = 0$
$A \oplus B = 0$	$x \oplus 0 = x$
$A = B$	from part a

Alternatively: $[A \oplus C = B \oplus C] \rightarrow (A = B) = (A \oplus C \oplus B \oplus C) + (A \oplus B)' = (A \oplus B) + (A \oplus B)' = 1$

c. $A \oplus B = A' \oplus B'$

Solution: $A \oplus B = A'B + AB' = AB' + A'B = (A')'B' + A'(B')' = A' \oplus B'$

d. $A \oplus (B + C) = (A \oplus B) + (A \oplus C)$

Solution:

False. Counter example: $A = C = 1, B = 0$

$\text{LHS} = 1 \oplus (0 \oplus 1) = 1 \oplus 1 = 0$

$\text{RHS} = (1 \oplus 0) + (1 \oplus 1) = 1 + 0 = 1$

3. [The containment relation] (15 Points)

Given two elements x and y of a Boolean algebra B such that $x \le y$, prove the truth of each of the following statements clearly stating your reasoning:

a. $xy' = 0$

b. $y' \le x'$

c. $xy = x$

Solution:

By definition, the partial ordering relation $x \le y$ implies that $x = \text{glb}(x, y) = xy$ and that $y = \text{lub}(x, y) = x + y$. This immediately proves (c). To prove (a), note that $xy' = (x)y' = (xy)y' = x(yy') = x(0) = 0$. To prove (b), we complement $x = xy$ and $y = x + y$ to obtain $x' = x' + y'$ and $y' = x'y'$ which, be definition, imply that $y' \le x'$.

4. [Cofactor and complement] (20 Points)

The cofactor of a function f with respect to an argument x (x') is the function that results from f when x is assigned the constant value 1(0) and is denoted by f_x ($f_{x'}$). Viewing cofactor as a unary operation show that cofactoring and complementation commute, i.e. show that:

$$(f')_x = (f_x)'$$

Illustrate the above identity with a function of your choice.

Solution:

$f = xf_x + x'f_{x'}$ (Boole's expansion theorem)

$f' = [x' + (f_x)'][x + (f_{x'})']$ (De Morgan's theorem)

$f' = x(f_x)' + x'(f_{x'})' + (f_x)'(f_{x'})' = x(f_x)' + x'(f_{x'})'$ (Multiply out and Consensus)

$$f' = x(f')_x + x'(f')_{x'}$$ (Shannon's expansion theorem applied to f')

The desired identity follows immediately by matching up the last two expressions for f'.

Example:

$$f = xy + z(x \oplus y) = xy + x'yz + xy'z$$ (Carry function of a full adder)

$$g \equiv f' = (x' + y')(x + y' + z')(x' + y + z')$$

$$f_x = y + y'z = y + z$$

$$(f_x)' = y'z'$$

$$g_x = (f')_x = y'(y + z') = y'z' = (f_x)'$$

5. [I like this one] (10 Points)

Given an n-variable (not necessarily 2-valued) Boolean function $f(x_1, \ldots, x_i, \ldots, x_n)$, determine the truth or falsehood of the following implication: $[f(x_1, \ldots, x_i, \ldots, x_n) = 0] \to [\forall x_i \cdot f(x_1, \ldots, x_i, \ldots, x_n) = 0]$. (*Hint: apply Boole's expansion.*)

Solution:

Given implication can be written as: $f' \to [\forall x_i \cdot f]'$

which is equivalent to: $f + \exists x_i \cdot f' = f + (f')_{\bar{x}_i} + (f')_{x_i}$

which on applying Boole's expansion w.r.t xi becomes:

$x_i \cdot f_{x_i} + x_i' \cdot f_{x_i'} + f'_{x_i} + f'_{x_i'} \equiv x_i + f'_{x_i} + x_i' + f'_{x_i'} = 1$

in the above expression first and third term, and 2nd and 4th term have been simplified using the expression x+x'y=x+y. As the expression evaluates to 1 so implication is true.

6. [Random problems from the textbook] (20 Points)

a. Problem 44, p. 120 in (H&S).(10 points)

One can easily see that

$$a + b = c + b \not\Rightarrow a = c, \tag{1}$$

and similarly that

$$a \cdot b = c \cdot b \not\Rightarrow a = c. \tag{2}$$

In the first case, $b = 1$ makes $a + b = c + b$, even though $a \neq c$. In the second case, a counterexample is obtained with $b = 0$.

Prove, *without using truth tables*, that

$$a + b = c + b \text{ and } a \cdot b = c \cdot b \Leftrightarrow a = c,$$

that is, if both (1) and (2) hold, then $a = c$.

[Hint: $x = y \Leftrightarrow x \oplus y = 0$.]

b. Problem 48, p. 121 in (H&S).(10 points)

Given $f = xy' + z$ and $d = xyz'$, find r such that f, r, and d describe an incompletely specified function, and compute $[l, u]$, the interval corresponding to that function.

Solution:

a. Given, $a + b = c + b$ (1)

and $a \cdot b = c \cdot b$ (2)

(1) can be written as $(a+b) \oplus (c+b) = 0$, after applying definition of XOR and applying De Morgan's Theorem this can be written as $b' \cdot (a \oplus c) = 0$ (3)

Similarly (2) can be written as $a \cdot b \oplus c \cdot b = 0$ after applying definition of XOR and applying De Morgan's Theorem this can be written as $b \cdot (a \oplus c) = 0$ (4)

ORing (3) and (4) we get,

$a \oplus c = 0 \Rightarrow a = c$.

An alternative derivation: instead of expanding the XORs, use the following identities of XORs:

$a + b = a \oplus b \oplus ab$, $ab \oplus bc = b(a \oplus c)$, $1 \oplus b = b'$ and $b \oplus b = 0$:

$$\begin{aligned}
(a+b = c+b) \text{ and } (ab = bc) &= [(a+b) \oplus (c+b) = 0] \wedge [(ab) \oplus (bc) = 0] \\
&= \{[(a+b) \oplus (c+b)] + [(ab) \oplus (bc)] = 0\} \\
&= \{[a \oplus b \oplus ab \oplus c \oplus b \oplus bc] + [b(a \oplus c)] = 0\} \\
&= \{[a(1 \oplus b) \oplus c(1 \oplus b)] + [b(a \oplus c)] = 0\} \\
&= \{b'(a \oplus c) + b(a \oplus c) = 0\} \\
&= \{a \oplus c = 0\} \\
&= \{a = c\}
\end{aligned}$$

b. $f = xy' + z$ and $d = xyz'$

In minterms representation, f and d can be written as:

$f = \{1, 3, 4, 5, 7\}$ and $d = \{6\}$

Therefore, in order to find r such that f, r and d describe an incompletely specified function, r should cover the rest of the minterms, i.e.

$r = \{0, 2\}$ => $r = x'y'z' + x'yz' = x'z'$

Alternatively, the relation between the on-set f, the off-set r, and the don't-care set d is that they partition the minterm space. Thus given any two, the third is the complement of their sum. We are given f and d, therefore: $r = (f + d)' = x'z'$.

Interval corresponding to the function is:

$[f, f + d] = [xy' + z, xy' + z + xyz'] = [xy' + z, x + z]$